Predicting elasticity in nonclastic rocks with a differential effective medium model

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ABSTRACT

Can a theoretical inclusion model — specifically, the differential effective medium (DEM) model — match experimental velocity data in rocks that are not necessarily made of inclusions, such as clastics? It is indeed possible in some cases by using an almost constant inclusion aspect ratio (AR) within wide ranges of porosity and mineralogy. We approach this question by using empirical velocity-porosity equations as proxies for data. By finding a DEM inclusion AR to match these equations, we find that the required range of AR is narrow. Moreover, a constant AR of about 0.13 can be used to accurately match empirical relations in competent sand, shale, and quartz/calcite mixtures. This finding can be utilized practically to predict $V_S$ from $V_P$; describe velocity-frequency dispersion between low-frequency and ultrasonic experiments; predict the dry-frame elastic properties from ultrasonic data on liquid-saturated samples where Gassmann’s fluid substitution is not applicable; predict the attenuation of P-wave velocity; and establish tight constraints for ranges of possible variation of $V_S$ and $V_P$ at a given porosity in some mineralogies. When we apply this approach to laboratory data rather than empirical equations, we confirm a positive answer to the main question, with all applications of this result still valid.

INTRODUCTION

Inclusion theories often represent porosity, pore geometry, and connectivity as distributions of ellipsoidal inclusions (Ament, 1953; Eshelby, 1957; Walsh, 1965; Wu, 1966; Kuster and Toksöz, 1974; O’Connell and Budiamsky, 1974; Berryman, 1980; Zimmerman, 1984, 1986, 1991a, 1991b; Cheng, 1993; Gurevich et al., 1998; Jakobsen et al., 2003b). These inclusion theories are based on extreme idealizations of the pore geometry and heuristic assumptions about the way in which high concentrations of pores interact elastically. The models offer useful analogs of the elastic behavior of some rocks with specific microstructures, but their limitation to idealized inclusion shapes complicates comparing the models to real rock microstructures. The shapes of rock pores are almost never ellipsoidal, and the methods for treating high concentrations of pores (inclusions) are idealized and heuristic. It is unrealistic to relate inclusion aspect ratio (AR) models to variations of rock texture that result from different depositional or diagenetic processes.

Inclusion models such as differential effective medium (DEM) (Norris, 1985) and the self-consistent approximation (Berryman, 1980) relate the elastic properties of rock to porosity (e.g., Xu and White, 1995), despite that some rocks are composed of grains rather than inclusions. The mathematical flexibility of inclusion theories enables such exercises because by varying the inclusion AR, it is possible to match almost any data point.

Therefore, we revisit this approach to use inclusion models, calibrated to measurements, as a single, flexible tool for implementing rock-physics transforms. Although many formulations (Kuster and Toksöz, 1974; O’Connell and Budiamsky, 1974; Berryman, 1980; Xu and White, 1995; Jakobsen et al., 2003a) can be used, we focus on the DEM approach as formulated by Norris (1985) with randomly oriented spheroid inclusions (Appendix A), though we deal with isotropic rock.

DEM can model different rocks with few adjustable parameters. Our approach consists of matching experimental data in reasonably wide ranges of porosity and mineralogy with DEM and finding the range of AR required to achieve this match. In other words, although the actual rock may not be like a physical realization of the mathematical model, we find the elastic equivalency between DEM and data. If elastic equivalency can be established with a narrow AR range for a wide range of real samples, we hypothesize that we can find an idealized physical analog to real rock and use the analog to interrelate different properties of its real prototype.

For pure quartz, quartz/clay, and quartz/calcite mineralogies, $V_P$ and $V_S$ can be accurately matched by DEM with AR $\approx 0.13$ in a porosity range from 0 to 0.4. This finding allows us to predict $V_S$ from $V_P$, quantify the effect of pore fluid on the velocity-frequency dispersion.
sion and assess attenuation, and derive constraints for the elastic properties narrower than the existing bounds.

We start by using the Raymer et al. (1980) empirical velocity-mineralogy-porosity model as a proxy for experimental data. We proceed by applying the concept to several laboratory data sets, which include ultrasonic velocity measurements from Coyer (1984), Rafavich et al. (1984), Han (1986), Kenter et al. (1997), Woodside et al. (1998), Asefa et al. (2003), Fabricius et al. (2008), and Verwer et al. (2008). The samples examined by these workers include siliciclastic, carbonate, and mixed carbonate-siliciclastic mineralogies. Finally, we demonstrate the utility of our findings on well logs from the Ocean Drilling Program (ODP) and a Venezuelan well (Graterol et al., 2004).

**DEM AND EMPIRICAL TRANSFORM**

The Raymer et al. (1980) relation links \( V_P \) to porosity \( \varphi \) as

\[
V_P = (1 - \varphi)^2 V_{Ps} + \varphi V_{Pf},
\]

where \( V_{Ps} \) is the velocity in the solid phase of the rock and \( V_{Pf} \) is the velocity in the fluid. In equation 1, the mineralogy affects \( V_{Ps} \) through \( V_{Ps} \). For example, \( V_{Ps} = 6.04 \text{ km/s in pure quartz, 4.51 km/s in a 50\% quartz and 50\% clay mixture, 6.22 km/s in a 50\% quartz and 50\% calcite mixture, and 6.64 km/s in pure calcite. Equation 1 was derived as an empirical fit to many log data points. Therefore, for our purposes, we can use it as experimental data. Although the equation originally was recommended for porosity of 0–0.37, we extend the range slightly to 0.40.

Figure 1. (Top row) P-wave velocity versus porosity according to Raymer et al. (1980) for five mineralogies. (Middle row) S-wave velocity according to Krief et al. (1990) as derived from the P-wave velocity in the top row. (Bottom row) AR derived by matching DEM/Gassmann to P-wave velocity (solid curve) and S-wave velocity (open circles).

**Figure 2. Velocity versus porosity for the five mineralogies examined in Figure 1. Solid curves are the same velocities as displayed in the top and middle rows of Figure 1. Symbols are the DEM/Gassmann predictions for a constant AR of 0.13. In the views where the solid curves are not apparent, they are overshadowed by the symbols.**

To extend the experimental data set into the \( V_S \) domain, we apply the Krief et al. (1990) \( V_S \) predictor to \( V_P \) derived from Raymer et al. (1980):

\[
\frac{V_P^2 - V_{Ps}^2}{V_S^2} = \frac{V_P^2 - V_{Pf}^2}{V_{Sf}^2}, \tag{2}
\]

where \( V_{Sf} \) is the shear-wave velocity in the solid phase.

In this example, we assume the rock is saturated with pure water with a density of 1.00 g/cm\(^3\) density and velocity of 1.50 km/s. Figure 1 displays the AR required to fit the data from the relations by DEM. When applying DEM, we assume that the rock is dry, find the dry-frame elastic moduli, and then use Gassmann’s (1951) fluid substitution to arrive at the results for water-saturated rock (DEM/Gassmann). For pure quartz, 50\% quartz and 50\% clay, and 50\% quartz and 50\% calcite mineralogies, the AR required to match these data is stable versus porosity — almost the same whether we find AR by matching DEM results to \( V_P \) or \( V_S \). Moreover, it is almost unchanged among these three mineralogies. For two other mineralogies (pure clay and pure calcite) examined in Figure 1, AR varies versus porosity and depends on whether it is determined from \( V_P \) or \( V_S \). Yet even in these two cases, AR is confined within a narrow range between 0.1 and 0.2.

We next select a single AR (e.g., AR = 0.13) and use it with DEM/Gassmann to predict \( V_P \) and \( V_S \) for all five mineralogies. For the three mineralogies where the DEM-derived AR remain almost constant (Figure 2), the predictions are very accurate. Even for the two mineralogies where AR varies versus porosity, the single AR (\( = 0.13 \)) can...
still be used to predict velocity with reasonable accuracy.

To use Gassmann’s equations, we must input the bulk moduli of the dry frame $K_{dry}$ and pore fluid $K_f$, respectively. For a multimineral rock matrix whose minerals have small elastic contrasts, the accuracy of Gassmann fluid substitution is adequate if the bulk modulus of the solid phase $K_s$ is computed using a mixing law, such as Hill’s (1952) average. For many rocks, especially those with high porosity, $K_s \gg K_{dry}$ and $K_s \gg K_f$. Therefore, $K_s$ may have a minor influence on the bulk modulus of the fluid-saturated rock $K_{sat}$, which allows application of Gassmann’s equation to multimineral rocks. For DEM modeling, $K_s$ is computed by the same approach.

We obtain comparable results when using other empirical relations as proxies for experimental data. For instance, we can estimate $V_S$ using the relation of Castagna et al. (1993) instead of that of Krief et al. (1990), or by Wyllie et al. (1956) combined with the $V_p$ by Krief et al. (1990) or by Castagna et al. (1993).

In Appendix B, we discuss predicting $V_S$ from $V_P$ using a self-consistent theory (Berryman, 1980) and compare the results to DEM predictions. In Appendix C, we examine the elastic equivalence approach using a soft-sand model (Dvorkin and Nur, 1996) and DEM.

DEM AND LABORATORY DATA

Consider a laboratory dry velocity data set that includes more than 60 siliciclastic samples with a wide range of porosity and clay content (Han, 1986). By using the earlier technique for the Raymer/Krief velocity equations, we find the AR that makes DEM/Gassmann match the data. In this case, we use the room-dry data and then apply Gassmann’s equation to obtain the velocity corresponding to water-saturated samples (low-frequency velocities). Once again, the AR fall within a narrow range, with an average value of approximately 0.13 (Figure 3).

Figure 3 shows data for chalk samples from four formations in the Danish central North Sea sector (Fabricius et al., 2008). As in the previous examples, we use data with subsequent water substitution. In this case, the DEM/Gassmann AR are approximately in the same interval as for Han’s (1986) data, although the mineralogy and porosity ranges differ. This fact suggests an approximate AR universality of about 0.13 for varying mineralogy and porosity. This universality is likely related to the texture of the data under examination. However, because we have fit real data with an idealized mathe-

Figure 3. P-wave velocity (left column), S-wave velocity (middle column), and AR (right column) versus porosity. In the right column, P-waves are denoted by open circles and S-waves are denoted by filled circles. The AR are obtained by matching the data by DEM/Gassmann. (a) Velocities at 40 MPa confining pressure, after Han (1986). (b) Velocities at 7.5 MPa confining pressure, after Fabricius et al. (2008); (c) Velocities at 50 MPa confining pressure, after Assefa et al. (2003). (d, e) Velocities at 40 MPa confining pressure, after Rafavich et al. (1984). The velocities are calculated from the room-dry experimental data using Gassmann’s (1951) fluid substitution. The fluid is pure water with density $\rho = 1.0 \text{ g/cm}^3$ and velocity $V = 1.5 \text{ km/s}$.
matical model, we do not overinterpret the meaning of this universality. Rather, we concentrate on its practical use.

Figure 3 also displays data from Assefa et al. (2003) for pure carbonates. As in the previous examples, we use room-dry data with subsequent water substitution. In this case, the DEM/Gassmann AR are smaller than in the previous examples. Still, they are contained within a narrow interval.

We next consider two laboratory data sets from Rafavich et al. (1984). These data sets are based on cores extracted from two different wells. The room-dry measurements were conducted at in situ pressure conditions (about 40 MPa). The data displayed in Figure 3 are essentially carbonate (dolomite and calcite). We use these room-dry data with subsequent water substitution. Once again, the DEM/Gassmann AR fall within a narrow interval, with 0.13 approximately in the middle (Figure 3).

As a second exercise, we use the same methodology for experimental wet velocity data on fully saturated rock samples. In this case, we find the AR that makes DEM match the velocity data at full saturation (high-frequency velocities). We consider two laboratory data sets (Figure 4). The first data set is from Kenter et al. (1997). These samples are mixed carbonate/siliciclastic mineralogy. Because of the presence of carbonate (see Figure 1 for the pure calcite case), the DEM/Gassmann AR span a wider range than in the previous example. The second data set is for wet carbonates, according to Woodside et al. (1998). Ultrasonic velocities, densities, and porosities were measured at in situ pressures in 68 plugs taken from cores recovered at ODP sites 966 and 967. The AR results are similar to those obtained for data from Kenter et al. (1997).

Rafavich (1984) and Assefa et al. (2003) visually measure the average pore size (longest and shortest pore diameter) on resin-impregnated polished thin sections. The aspect ratios are measured only on pores large enough (>20 μm) to be seen through an optical microscope. This analysis was performed mainly for the grainstone and packstone subsets of the data. The AR of the small pores, which constitute up to 10% of the total pore volume, are not measured (Assefa et al., 2003). The AR reported by Rafavich (1984) span an interval of 0.4–1.0. The AR reported by Assefa et al. (2003) are about 0.25. These AR are much larger than those predicted by our DEM/Gassmann method.

Figure 4. P-wave velocity (left column), S-wave velocity (middle column), and AR (right column) versus porosity. In the right column, P-waves are denoted by open circles and S-waves are denoted by filled circles. (Top row) Velocities at 40 MPa confining pressure, after Kenter et al. (1997). (Bottom row) Velocities at 7.5 MPa confining pressure, after Woodside et al. (1998).

This disparity emphasizes the fact that AR obtained from a mathematical model often differ from those in real rock (even if we assume that pores in real rock can be treated as ideal shapes). Indeed, no mathematical model can mimic real rock perfectly. Therefore, it can be erroneous to derive real pore-space geometry from idealized models. Nevertheless, a model can be useful and predictive, as we show in the following sections.

APPLICATIONS

Predicting \( V_S \)

The methodology, when dry velocity data are available (Figure 3), is to obtain the AR for each \( V_P \) data point by matching it with DEM/Gassmann and then to use DEM/Gassmann with the same AR to calculate \( V_S \) for the same data point. These AR are then used to determine the dry bulk \( K_{dy} \) and shear \( G \) moduli. When wet velocity data are available (Figure 4), the methodology is to obtain the AR for each \( V_S \) data point by matching it with DEM alone and then to use DEM with the same AR to calculate \( V_S \) for the same data point. These AR are then used to determine the wet bulk \( K_{sw} \) and \( G \).

The results for Han’s (1986) data are displayed in Figure 5, where the dry elastic moduli and \( V_S \) are predicted with a high degree of accuracy. Similar accuracy is achieved for data from Fabricius et al. (2008) and Rafavich et al. (1984) (Figure 5). For the data from Assefa et al. (2003), we overpredict the measured \( V_S \) if we directly use the AR derived from \( V_P \) (Figure 5). We have found an ad hoc correction for pure carbonate cases: When predicting \( V_S \), use the AR derived from \( V_P \) and multiply it by 0.85. For the data from Kenter et al. (1997), the wet elastic moduli and the resulting \( V_S \) appear accurate (Figure 6).

The next two examples are from well-log data from a Venezuelan oil field and the ODP. Our \( V_P \) prediction methodology works well for the Venezuelan well, which is mostly siliciclastic (Figure 7). However, \( V_S \) is overpredicted in the ODP well (Figure 8) when we directly use the AR derived from \( V_P \). Our ad hoc correction (the multiplication of the \( V_P \)-derived AR by 0.85) accurately predicts \( V_S \) in this carbonate well.
Figure 5. Dry bulk modulus (left column), shear modulus (middle column), and $V_s$ prediction (right column) for data from (a) Han (1986), (b) Fabricius et al. (2008), (c) Assefa et al. (2003), and (d, e) Rafavich et al. (1984). The velocities are calculated from the room-dry experimental data using Gassmann’s (1951) fluid substitution.

Figure 6. (a) Dry bulk and (b) shear moduli and (c) wet $V_s$ prediction for Kenter’s et al. (1997) data. The dry moduli and wet velocities are calculated from the room-wet experimental data.
Figure 7. $V_s$ prediction for the Venezuelan well: (a) mineralogy and porosity, (b) measured (black) and predicted (red) $V_s$, and (c) AR derived from $V_p$ using DEM/Gassmann.

Figure 8. $V_s$ prediction for the ODP site 1172 well. (a) Mineralogy and porosity. (b) The blue $V_s$ curve is the prediction directly using the AR derived from $V_p$. The red $V_s$ curve is predicted using the $V_p$-derived AR multiplied by 0.85. (c) AR derived from $V_p$ using DEM/Gassmann.
Velocity-frequency dispersion

Velocity-frequency dispersion was discovered decades ago when it manifested itself as a disparity between the velocity measured on wet samples at a very high frequency \( f_H \) (on the order of 1 MHz) and at low frequency \( f_L \) calculated from dry-rock data using Gassmann’s (1951) fluid substitution (e.g., Mavko and Jizba, 1991). The velocity measured at low frequency is usually smaller than that measured at high frequency because at high frequency, the liquid in the pores is unrelaxed and resists the wave-induced deformation essentially as an elastic body.

DEM offers a simple recipe to calculate this difference in velocity. For example, to predict the velocity at high frequency from that measured at low frequency, we obtain an AR for each \( V_P \) and \( V_S \) data point (separately for \( V_P \) and \( V_S \)) by matching it with DEM/Gassmann. We then use DEM with these AR and with the liquid in the pores treated as an elastic body to calculate the high-frequency \( V_P \) and \( V_S \) for the same data point. Conversely, to predict the low-frequency velocity from high-frequency data, we obtain the AR for each \( V_P \) and \( V_S \) data point (separately for \( V_P \) and \( V_S \)) by matching it with DEM, which treats the liquid in the inclusions as an elastic body. Then we use DEM/Gassmann with the same AR to obtain the low-frequency \( V_P \) and \( V_S \).

As an example of this transformation, in Figure 9 we display data from Coyner (1984), with the velocity measured on room-dry and water-saturated samples of Navajo Sandstone. Here, we predict the high-frequency velocity using the room-dry data. In Figure 10, we show how low-frequency velocity can be obtained from ultrasonic wet velocity for the same data. In both cases, our predictions are close to the measurements. Our goal is not to predict the pressure dependency of \( V_P \) and \( V_S \) at low and high frequencies but to predict velocities at fixed pressure. To achieve this goal, we adjust the AR so the theory fits the velocity measurements at fixed pressure. The computed AR increase with the increasing pressure because the rock becomes stiffer (Figure 9).

For velocity predictions at all frequencies (Dvorkin et al., 1995), a model with a spectrum of AR, instead of a single one, is a better analog of rock porosity and pore shapes (Hudson et al., 1996; Chapman et al., 2002, Jakobsen et al., 2003b; Jakobsen and Johansen, 2005); however, this calculation is beyond the scope of our study.

Empirical constraints for velocity

Rigorous elastic bounds for velocity, as provided by Hashin and Shtrikman (1963), are typically far apart for a porous rock. This fact limits the practical use of these bounds to estimate velocity from porosity and mineralogy.

Our recipe for constraining the velocity is threefold: (1) find the elastic moduli and density of the solid (mineral) phase by using, e.g., Hill’s (1952) average; (2) compute the lower constraints for \( V_P \) and \( V_S \) from DEM/Gassmann using \( AR = 0.03 \); and (3) compute the upper constraints using \( AR = 0.30 \).

If we deal with ultrasonic data obtained on wet samples, DEM/Gassmann may be replaced by pure DEM where the pore fluid is treated as an elastic inclusion. Figure 11 shows an example of these constraints for the Kenter et al. (1997) ultrasonic wet-rock data. The DEM-derived constraints are much tighter than Hashin-Shtrikman (1963) bounds, especially for \( V_S \). Of course, AR of 0.03 and 0.30 are selected ad hoc. The values may differ for other data sets. Our rec-

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**Figure 9.** (a) \( V_P \) versus confining pressure. Open circles are for room-dry ultrasonic data; filled circles are for wet ultrasonic data. The continuous curve is for a low-frequency wet sample obtained by Gassmann’s (1951) fluid substitution on the room-dry data. Triangles are for high-frequency velocity predicted from room-dry data using DEM. (b) Same for \( V_S \). (c) AR predicted using DEM on the room-dry data for \( V_P \) (filled circles) and \( V_S \) (open circles). The triangles in (a) are obtained using the \( V_P, AR \).

**Figure 10.** (a) \( V_P \) versus confining pressure. Open circles are for room-dry ultrasonic data; filled circles are for wet ultrasonic data. The continuous curve is for a dry-rock velocity obtained from the ultrasonic data using Gassmann’s (1951) fluid substitution from wet to dry (which is erroneous in this context because the wet-rock data come from ultrasonic measurements). Triangles are a prediction for room-dry velocity using our methodology. (b) Same for \( V_S \). (c) AR predicted using DEM on the ultrasonic data for \( V_P \) (filled circles) and \( V_S \) (open circles). The triangles displayed in (a) are obtained using the \( V_P, AR \).
ommendation is to establish the optimal range of AR on a typical test subset and then use it for the same rock type on a wider basis.

An example of such an approach is given in Figure 12, where we use AR = 0.1 and 0.3 to constrain the velocity in a tight-sand well. As seen, the DEM-based constraints are much tighter than the Hashin-Shtrikman bounds.

**DISCUSSION**

If treated and interpreted properly, an idealized elastic model such as DEM can be predictive in real rock, in a practical sense, without overgeneralizing the model’s meaning and utility. The latter is criti-

ical in overgeneralization applications. Indeed, we are attempting to match the overwhelming complexity of rock behavior with a single AR (or a single range of AR). When doing so, any generalization must be based on meaningful examples, such as shown above. Over-

extending these results may be erroneous.

Consider, for example, a data set by Verwer et al. (2008), consisting of dolomite/calcite samples. As we see, DEM applied to ultrasonic wet-rock data predicts AR as high as 0.4 — even approaching 1.0 in the samples (Figure 13). The values are much higher than 0.13, which is the value almost universally used in our analysis. This means we must always tread cautiously when applying mathematical models to real data. Such models may be relevant and useful for a certain type of rock but may be completely invalid for another type of sediment. Figure 13 shows that although the AR range in the Verwer et al. (2008) data is quite different from that used in most examples here, it can still be used successfully to predict $V_S$ from $V_P$.

Figure 14 shows that rock is not made of ideally shaped inclusions. The main point of our work is that certain effective-medium inclusion models can mimic the elastic properties of real rock if an appropriate AR is assigned to the model. Our goal is to make the use of a selected model reasonably general, rather than insist that the geometry of the model is the same as of a real rock. Moreover, the AR is merely a fitting parameter within the realm of a selected model.

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**Figure 11.** The Hashin-Shtrikman (1963) bounds (gray curves) and DEM-based constraints (black curves) for the Kenter et al. (1997) data (circles). (a) for $V_P$ and (b) $V_S$.

**Figure 12.** The Hashin-Shtrikman (1963) and DEM-based constraints for (a) $V_P$ and (b) $V_S$ in a Venezuelan well (different depth interval of the well in Figure 7). The AR range used in the DEM constraints is between 0.10 and 0.30. The white curve is the measured velocity.

**Figure 13.** (a) $V_P$ and (c) $V_S$ versus porosity (data from Verwer et al., 2008). (b) DEM aspect ratio predicted from $V_P$ and $V_S$. (d) $V_S$ predicted from $V_P$ using these AR and our methodology.

**Figure 14.** 2D slices of a 3D CT scan of Fontainebleau Sandstone. No ideal inclusions are present. AR = 0.13 does not make sense. The pictures show that rock is not made of idealized inclusions; such inclusions can be assumed in a selected model to match elastic data.
Conclusions

Given a rock with known porosity and mineralogy and a complex microstructure, an idealized effective-medium model can represent the elastic properties with a narrow range of AR. This inclusion AR range required for a differential effective-medium model to match measured velocity is narrow for many data sets, including laboratory and well data. Although this finding is practical, we caution against overextending the range of its applications without verifying and calibrating the model against a representative data set.

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Appendix A

Dem Theory

Dem theory assumes that a composite material may be constructed by making infinitesimal changes in an already existing composite. If the effective bulk and shear constants of the composite are $K^*(y)$ and $G^*(y)$, where the volume fraction of the inclusion phase is $y$, the equations governing the changes in these constants are (Mavko et al., 2009)

\[
(1 - y) \frac{d}{dy}[K^*(y)] = (K_2 - K^*)P^{(y2)}(y),
\]

\[
(1 - y) \frac{d}{dy}[G^*(y)] = (G_2 - G^*)Q^{(y2)}(y),
\]

with initial conditions $K^*(0) = K_1$ and $G^*(0) = G_1$, where $K_2$ and $G_2$ are the bulk and shear moduli of the initial host material, respectively, and $K_1$ and $G_1$ are the bulk and shear moduli of the incrementally added inclusions, respectively. In porous rock, $y$ is the total porosity $\phi$. The coefficients $P$ and $Q$ are geometric factors dependent upon the shape of the inclusion (Mavko et al., 2009). Here, we use the ellipsoidal inclusions.

The superscript $*2$ for $P$ and $Q$ indicates the factors are for the inclusions. The subscript $*$ is for the background medium whose bulk modulus is $K^*$ and shear modulus is $G^*$. Fluid-saturated cavities are simulated by setting the inclusion shear modulus to zero.

The coefficients $P$ and $Q$ for ellipsoidal inclusions of an arbitrary AR are given by

\[
P = \frac{1}{2} T_{ijij}^* \quad Q = \frac{1}{2} \left( T_{ijij}^* - \frac{1}{3} T_{ijij}^2 \right),
\]

where the tensor $T$ relates the uniform far-field strain to the strain within the ellipsoidal inclusion (Wu, 1966). In this study, the elastic properties of the inclusions are set as those of seawater at in situ conditions; the matrix properties are for pure calcite.

Appendix B

Elastic Equivalency Using A Self-Consistent Model

In this appendix, we compare predicting $V_p$ based on $V_p$ using a self-consistent (SC) theory known as the coherent potential approximation (Berryman, 1980) with DEM prediction (Norris, 1985). These models represent porosity and pore shape as ellipsoidal inclusions with different AR.

DEM assumes isolated pores embedded in a host material that remains continuous at all porosities. SC treats grains and pores symmetrically; instead of requiring a single background material, grains and pores can be connected or disconnected, depending on the porosity range. Both models are physically realizable and offer useful analogs of the elastic behavior of some rocks with specific microstructures. For instance, rocks composed of grains look more like a physical realization of SC than DEM (Berge et al., 1995), and rocks with vugs, such as some carbonates, look more like a physical realization of DEM. For rocks with the same solid and fluid constituents, the two models produce different results, i.e., it is necessary to use SC aspect ratios significantly greater than the DEM aspect ratios to match the same velocity. Consequently, the interpretation of rock-pore AR may not be unique. Moreover, the AR is just a fitting parameter within the realm of a selected model.

Figure B-1 shows the $V_p$ and $V_s$ velocities versus porosity according to an SC model for wet calcite. The SC velocities are modeled using AR of 1.0 and 0.25 for the mineral and the pores, respectively, and are kept constant in the entire porosity range. We compute the SC aspect ratios by matching DEM to the SC $V_p$ and $V_s$ values. The computed DEM aspect ratios are lower than the SC aspect ratios in the entire porosity range, and the differences increase with porosity. The AR to match $V_p$ (DEM ARp) and $V_s$ (DEM ARs) are almost equal. Similar results are obtained modeling SC velocities when using lower AR for the solid constituents and when using different minerals.

As a second exercise, we repeat the example shown in Figure 1 but use the SC model instead of DEM. Figure B-2 shows the AR required to fit the velocity data from SC by DEM. The velocities from SC are computed assuming a constant AR of 1.0 for the solid mineral and 0.25 for the pores. For all mineralogies, the SC aspect ratios re-
required to match Raymer’s data show a greater variation with porosity than those computed using DEM (Figure B-2). For pure quartz, 50% quartz and 50% clay, and 50% quartz and 50% calcite, the calculated SC aspect ratio depends only slightly on whether it was determined from $V_P$ or $V_S$. For pure clay and calcite, the SC aspect ratio depends more strongly on whether it was determined from $V_P$ or $V_S$.

If the SC aspect ratios required to match the Raymer et al. (1980) $V_P$ data are used to predict $V_S$ using SC/Gassmann again, the $V_S$ prediction is accurate and computationally faster than DEM; however, the SC aspect ratios vary considerably with porosity (Figure B-3).

Figure B-1. (a) P- and S-wave velocity versus porosity according to SC for wet calcite. The AR used to model the SC velocities are constant and equal 1.0 for the mineral and 0.25 for the pores. (b) AR derived by matching DEM to the SC $P$-wave (filled circles) and S-wave (open circles) velocities. The same AR is observed for different minerals.

Figure B-2. (Top row) P-wave velocity versus porosity according to Raymer et al. (1980) for five mineralogies. (Middle row) S-wave velocity according to Krief et al. (1990) as derived from the P-wave velocity in the top row. (Bottom row) AR derived by matching SC/Gassmann to P-wave velocity (solid curve) and S-wave velocity (circles).

Figure B-3. Velocity versus porosity for the five mineralogies examined in Figure B-2. Solid curves are the same velocities as displayed in the top and middle rows of Figure B-2. Symbols are the SC/Gassmann S-wave velocity predictions for the AR derived from SC by matching SC/Gassmann to the P-wave velocity according to Raymer et al. (1980).
APPENDIX C

ELASTIC EQUIVALENCY USING DEM AND GRANULAR MODELS

In this appendix, the equivalent elastic model concept is applied to granular-medium models. The velocities determined using granular-medium models are matched with the velocities determined directly using DEM in wet rock, treating the fluid in the pores as an elastic inclusion (high-frequency estimate) as well as a combination of DEM for dry rock with subsequent Gassmann’s fluid substitution (low-frequency estimate). For this exercise, two heuristic, granular-medium models are used: soft sand (Dvorkin and Nur, 1996) and stiff sand (Gal et al., 1998). We explore the elastic equivalency between DEM and these models, even though the actual rocks may not be like a physical realization of the soft- and stiff-sand models.

The soft-sand model has been used successfully to model deep-ocean shallow buried sediments (Dvorkin and Prasad, 1999; Prasad and Dvorkin, 2001). This model is based on the Hertz-Mindlin theory (Mindlin, 1949) and the modified Hashin-Strikman lower bound. The contact Hertz-Mindlin theory gives expressions for the effective bulk and shear moduli of a dry, dense, random pack of identical spherical grains subject to a hydrostatic pressure $P$.

To find the effective bulk $K_{\text{dry}}$, and shear $G_{\text{dry}}$, moduli at a different porosity, Dvorkin and Nur (1996) propose a heuristic modified Hashin-Strikman lower bound based on the original Hashin-Strikman lower bound (1963):

$$K_{\text{dry}} = \left( \frac{\varphi}{\varphi_c} + \frac{1 - \varphi}{K_{\text{HM}} + \frac{4}{3}G_{\text{HM}}} \right)^{-1} - \frac{4}{3}G_{\text{HM}},$$

$$G_{\text{dry}} = \left( \frac{\varphi}{\varphi_c} + \frac{1 - \varphi}{G_{\text{HM}} + Z_1 + G + Z_1} \right)^{-1} - Z_1,$$

$$Z_1 = \frac{3K_{\text{HM}} + 8G_{\text{HM}}}{K_{\text{HM}} + 2G_{\text{HM}}},$$

where $\varphi_c$ is the critical porosity and $K$ and $G$ are the elastic moduli of the solid phase.

This model connects two end members — one with zero porosity and the moduli of the solid phase, and the other with critical porosity and pressure-dependent moduli as given by the Hertz-Mindlin theory:

$$K_{\text{HM}} = \frac{3C^2(1 - \varphi_c)^2G^2}{18 \pi^2(1 - \nu)^2P},$$

$$G_{\text{HM}} = \frac{5 - 4\nu}{5(2 - \nu)} \frac{3C^2(1 - \varphi_c)^2G^2}{2 \pi^2(1 - \nu)^2P},$$

where $\nu$ is the Poisson’s ratio of the porous grains, $P$ is the differential pressure acting upon the pack, and $C$ is the average number of contacts that each grain has with its neighboring grains (the coordination number).

To determine the effective velocities for low-frequency water-saturated rock, Gassmann’s (1951) formula is used. Following Gassmann’s (1951) fluid substitution equations, $K_{\text{dry}}$ and shear $G_{\text{dry}}$ are given by

$$\varphi K_{\text{dry}} - (1 + \varphi)K_f \frac{K_{\text{dry}}}{K} + K_f,$$

$$K_{\text{sat}} = K - (1 - \varphi)K_f + \varphi K - K_f \frac{K_{\text{dry}}}{K},$$

$$G_{\text{sat}} = G_{\text{dry}},$$

where $K_f$ is the bulk modulus of water.

Finally, the elastic $P$- and $S$-wave velocities and the bulk density $\rho_b$ are

$$V_p = \sqrt{\frac{K_{\text{sat}} + \frac{4}{3}G_{\text{sat}}}{\rho_b}},$$

$$V_S = \sqrt{\frac{G_{\text{sat}}}{\rho_b}},$$

and $\rho_b = (1 - \varphi)\rho_s + \varphi\rho_f$.

The elastic equivalency between the soft-sand and DEM models is achieved by finding the AR required to match the dry or wet soft-sand elastic moduli with those provided by DEM, where the pore fluid is treated as an elastic inclusion (high-frequency estimate) and the DEM-Gassmann combination (low-frequency estimate), respectively (Figure C-1).

Figure C-2 shows the AR required for fitting the velocities from the soft-sand model by the DEM/Gassmann combination. We assume the rock is saturated with pure water with 1.00 g/cm$^3$ density and 1.50 km/s velocity. The input parameters for the soft-sand model are $K = 37$ GPa, $G = 44$ GPa, $K_f = 2.25$ GPa, $P = 10$ MPa, $C = 9$, and $\varphi_c = 0.44$ (as used in Prasad and Dvorkin, 1999). In the DEM model, we use the same values for $K$, $G$, and $K_f$ and vary the AR to fit the soft-sand model results.

When modeling with DEM/Gassmann and the soft-sand model, we assume the rock is dry, find the dry-frame elastic moduli, and then use Gassmann’s (1951) fluid substitution to arrive at the results for water-saturated rock. For all mineralogies, the AR required to match these data increases linearly with increasing porosity (Figure C-2). In addition, the AR required are small and confined within a range of approximately 0.02–0.1 in the entire porosity range. The most important result is that the AR at all porosities are independent of whether they were determined from $V_p$ or $V_S$.

In the soft-sand model, $C$ and $P$ are kept constant. If any of these parameters increases, the AR needed to match the soft-sand model
The technique discussed in the main text allows us to find the high-frequency velocities from low-frequency velocities. To achieve this, we use the AR obtained in the low-frequency range and compute the elastic moduli for these inclusions using DEM alone, rather than the DEM/Gassmann combination. The results are displayed in Figure C-3.

Let us reiterate that the AR obtained using this fitting technique are not sensitive to whether they are determined from $V_p$ or $V_S$. Based only on $V_p$, these results allow us to estimate (a) $V_S$, (b) high-frequency wet-rock velocity from dry-rock data (and vice versa), and (c) maximum $P$- and $S$-wave attenuation.

REFERENCES

Han, D., 1986, Effects of porosity and clay content on acoustic properties of sandstones and unconsolidated sediments: Ph.D. dissertation, Stanford University.
Predicting elastic properties using DEM


